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The q-Deformed Vector and q-Deformed Outer Product

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In this paper the q-deformed vector is introduced and the q-deformed outer product is investigated.

Since Manin (1987, 1988, 1989) proposed the idea of the quantum plane, much has been developed in this direction. Woronowicz (1987, 1989) applied noncommutative differential geometry to the quantum matrix group. Wess and Zumino (1990; Zumino, 1991) considered one of the simplest examples of noncommutative differential calculus over Manin's plane. They developed a differential calculus over the quantum hyperplane covariant with respect to the action of the quantum deformation of $GL(n)$ which is called $GL_q(n)$. Much work has followed in this direction (Schmidke *et al.*, 1989; Schirrmacher, 1991a,b; Schirrmacher *et al.*, 1991; Burdik and Hlavaty, 1991; Hlavaty, 1991; Burdik and Hellinger, 1992; Giler *et al.*, 1991; Ewen, 1991; Ge *et al.*, 1992; Kupershmidt, 1992; Brzezinski, 1992, 1993; Brzezinski *et al.*, 1992; Aref'eva and Volovich, 1991; Schwenk and Wess, 1992).

In this paper we define the q-vector and q-deformed outer product and investigate their properties by using the q-deformed Levi-Civita symbol and its properties (Chung *et al.*, 1994).

We define the q-vectors **A** and **B** in the three-dimensional quantum hyperplane as

$$\mathbf{A} = (A_1, A_2, A_3) \tag{1}$$

$$\mathbf{B} = (B_1, B_2, B_3) \tag{2}$$

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which satisfy the condition of q-vectors

$$A_iA_j = q_{ij}A_jA_i \tag{3}$$

$$B_iB_j = p_{ij}B_jB_i \tag{4}$$

$$[A_i, B_j] = 0 \qquad \text{for all} \quad i, j = 1, 2, 3 \tag{5}$$

Here we demand that

$$q_{ji} = \frac{1}{q_{ij}}, \qquad q_{ii} = 1$$

$$p_{ji} = \frac{1}{p_{ij}}, \qquad p_{ii} = 1$$

Then we define the q-deformed outer product of two q-vectors **A** and **B** as follows:

$$(\mathbf{A} \times \mathbf{B})_i = E_{ijk}A_jB_k \tag{6}$$

We require that the q-deformed outer product remain a q-vector, which implies that **A** × **B** should satisfy a similar commutation relation to (3), (4):

$$(\mathbf{A} \times \mathbf{B})_i(\mathbf{A} \times \mathbf{B})_j = r_{ij}(\mathbf{A} \times \mathbf{B})_j(\mathbf{A} \times \mathbf{B})_i \tag{7}$$

where r_{ij} satisfies

$$r_{ji} = \frac{1}{r_{ij}}, \qquad r_{ii} = 1$$

For all i and j, equation (7) gives the following consistency conditions:

$$\begin{aligned} r_{12}p_{12} &= 1, & r_{12}q_{12} &= 1 \\ r_{12}p_{13} &= q_{23}, & r_{12}q_{13} &= p_{23} \\ r_{23}p_{23} &= 1, & r_{23}q_{23} &= 1 \\ r_{23}p_{13} &= q_{12}, & r_{23}q_{13} &= p_{12} \\ r_{13}p_{13} &= 1, & r_{13}q_{13} &= 1 \\ r_{13}q_{12}p_{23} &= 1, & r_{13}q_{23}p_{13} &= 1 \end{aligned} \tag{8}$$

Equations (8) yield the condition

$$p_{ij} = q_{ij} \qquad \text{for all} \quad i, j = 1, 2, 3 \tag{9}$$

Using equation (8), we have

$$\begin{aligned} r_{12}q_{12} &= 1, & r_{23}q_{23} &= 1, & r_{13}q_{13} &= 1 \\ r_{12}q_{13} &= q_{23}, & r_{23}q_{13} &= q_{12}, & r_{13}q_{12}q_{23} &= 1 \end{aligned} \tag{10}$$

Eliminating all r_{ij} in (10), we have the following relation among three independent q_{ij}:

$$q_{13} = q_{12}q_{23} \tag{11}$$

In particular, if we set

$$q_{12} = q_{23} = q$$

we determine all the deformation parameters in terms of q as follows:

$$\begin{aligned} q_{ij} &= p_{ij} = q^{j-i} \\ r_{ij} &= q^{i-j} \end{aligned} \tag{12}$$

To sum up, for two q-vectors **A** and **B** we have

$$\begin{aligned} A_iA_j &= q^{j-i}A_jA_i \\ B_iB_j &= q^{j-i}B_jB_i \end{aligned} \tag{13}$$

Henceforth we call a vector satisfying (13) a q-vector. Then the q-deformed outer product of two q-vectors **A** and **B** satisfies the following commutation relation:

$$(\mathbf{A} \times \mathbf{B})_i(\mathbf{A} \times \mathbf{B})_j = q^{i-j}(\mathbf{A} \times \mathbf{B})_j(\mathbf{A} \times \mathbf{B})_i \tag{14}$$

Equation (14) is easily proved by using the definition of q-vectors.

Let us consider the relation

$$(\mathbf{A} \times \mathbf{B})_i(\mathbf{A} \times \mathbf{B})_j = r_{ij}(\mathbf{A} \times \mathbf{B})_j(\mathbf{A} \times \mathbf{B})_i \tag{15}$$

We can write the left-hand side of (15) as

$$\begin{aligned} \text{LHS} &= E_{ilm}A_lB_mE_{jst}A_sB_t \\ &= E_{ilm}E_{jst}A_lA_sB_mB_t \\ &= E_{ilm}E_{jst}q^{s-l}q^{t-m}A_sA_lB_tB_m \end{aligned} \tag{16}$$

Similarly, the right-hand side of (15) can be written as

$$\text{RHS} = r_{ij}E_{jst}E_{ilm}A_sA_lB_tB_m \tag{17}$$

Comparing (16) with (17), we obtain

$$r_{ij} = q^{s-l+t-m} \qquad \text{for all} \quad s, l, t, m = 1, 2, 3 \tag{18}$$

where $l \neq m \neq i$ and $s \neq t \neq j$. Since $i + l + m = j + s + t = 6$, we find

$$r_{ij} = q^{i-j}$$

which completes the proof of (14).

We call **A** a q-vector when the following relation is satisfied:

$$A_i A_j = q^{j-i} A_j A_i \tag{19}$$

Similarly, let us call **B** a q^{-1}-vector when the following condition is satisfied:

$$B_i B_j = q^{i-j} B_j B_i \tag{20}$$

Then we can remark that when both **B** and **C** are q-vectors the q-deformed outer product of **A** and **B**, **A** × **B**, is a q^{-1}-vector. When both **B** and **C** are q-vectors, **A** should be a q^{-1}-vector so that the q-deformed outer product **A** × (**B** × **C**) is well defined. Then the final deformed vector becomes a q-vector. Therefore we can say that the q-deformed outer product transforms a q-vector into a q^{-1}-vector and vice versa. In this paper we deformed the three-dimensional vector and the outer product. We obtained a special condition for the q-vector so that the q-deformed outer product is well defined. Much remains to be done in this direction. First, the q-deformed inner product must be well defined for the complete theory of q-deformed vectors. Second, we should generalize the concept of a q-deformed vector to a q-deformed tensor theory in an arbitrary-dimensional quantum hyperplane. We hope that these problems and related topics will be clarified in the near future.

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